Lecture 5, 10/08/12

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Matrix Algebra

- Matrix / matrix multiplication
- Matrix / number multiplication
- Matrix / matrix addition and subtraction.
- Matrix division (maybe on Wed.)

Matrix Multiplication Continued

- Let A be m_1 x n_1 and B be m_2 x n_2.
  - Then $A \cdot B$ only makes sense if $n_1 = m_2$.
  - The result is a matrix with dimensions m_1 x n_2

$$m_1 \times \boxed{n_1 \qquad m_2} \times n_2$$

$$m_1 \times n_2$$

Matrix Multiplication

$$\begin{array}{cc} A & B \\ 3 \times \boxed{3 \quad\quad 3} \times 2 & \\ \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix} \cdot & \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ -1 & 0 \end{bmatrix} \end{array}$$

$$A \cdot B = \begin{bmatrix} \text{R1} * \text{C1} & \text{R1} * \text{C2} \\ \text{R2} * \text{C1} & \text{R2} * \text{C2} \\ \text{R3} * \text{C1} & \text{R3} * \text{C2} \end{bmatrix}$$

Matrix Multiplication

$$\begin{array}{ccc} A & & B \\ 3 \times \boxed{3} & & \boxed{3} \times 2 \end{array}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ -1 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2 \cdot 1 + 2 \cdot (-3) + (-1) \cdot (-1) & \text{R1} * \text{C2} \\ \text{R2} * \text{C1} & \text{R2} * \text{C2} \\ \text{R3} * \text{C1} & \text{R3} * \text{C2} \end{bmatrix}$$

Matrix Multiplication

- Be careful. Matrix multiplication does not commute.

$$A \cdot B \neq B \cdot A$$

Matrix / Vector Formulation of a System

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ -2 \\ 16 \end{bmatrix}}_{\vec{b}}$$

$$\Updownarrow$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 3 \\ -2x_2 + 3x_3 &= -2 \\ x_1 + 5x_3 &= 16 \end{aligned}$$

Notation and Conventions

$$A \cdot \vec{x} = \vec{b}$$

- A - is referred to as the coefficient matrix.
- $\vec{\square}$ - is a notation indicating a vector
- $\vec{x}$ - is called the vector of unknowns
- $\vec{b}$ - is called the constant vector

Matrix Algebra Continued

Matrix \ Number Multiplication

$$\# \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} \# \cdot 2 & \# \cdot 1 & \# \cdot -1 \\ \# \cdot 0 & \# \cdot -2 & \# \cdot 3 \\ \# \cdot 1 & \# \cdot 0 & \# \cdot 5 \end{bmatrix}$$

- Exactly what you would think it is.
  - No restriction on dimension of matrix.

Example

$$12 \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 12 \cdot 2 & 12 \cdot 1 & 12 \cdot -1 \\ 12 \cdot 0 & 12 \cdot -2 & 12 \cdot 3 \\ 12 \cdot 1 & 12 \cdot 0 & 12 \cdot 5 \end{bmatrix}$$

Matrix Addition

- You can add two matrices <u>only if they are the exact same size</u>.
- Let A and B both be m x n.

$$\begin{bmatrix} 0 & -3 & 2 \\ 5 & -4 & 6 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 1 \\ 0 & 1 & 0 \\ 3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 0+(-2) & -3+4 & 2+1 \\ 5+0 & -4+1 & 6+0 \\ 1+3 & -1+6 & 2+(-2) \end{bmatrix}$$

What we've covered

- Matrix / matrix multiplication
- Matrix / matrix addition
- Matrix / number multiplication

What's Left?

- Matrix "division".
- Division is really just multiplication by an ***inverse***.

$$\frac{5}{3} = \frac{1}{3} \cdot 5 = 3^{-1} \cdot 5$$

- You would call **1/3** the inverse of **3**.

What is an inverse?

- The number you multiply by to get 1.

$$2^{-1} = \frac{1}{2} \longrightarrow 2 \cdot \frac{1}{2} = 1$$

$$(-6)^{-1} = \frac{1}{-6} \longrightarrow (-6) \cdot \frac{1}{-6} = 1$$

$$(\pi)^{-1} = \frac{1}{\pi} \longrightarrow (\pi) \cdot \frac{1}{\pi} = 1$$

The Identity

- In regular algebra / arithmetic , 1 is referred to as the ***identity***.
- Multiplying by it does not change anything.

$$z \cdot 1 = z$$

Identity Matrix

- We need to first define the concept of an ***identity matrix***
- Define

$$\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Then for any matrix A that is the proper size $A \cdot \mathbb{I} = A = \mathbb{I} \cdot A$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 & -2 & 0 \\ -3 & 16 & 7 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 0 \\ -3 & 16 & 7 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -2 & 0 \\ -3 & 16 & 7 \\ 0 & 4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 0 \\ -3 & 16 & 7 \\ 0 & 4 & -1 \end{bmatrix}$$

Identity Matrix

- The ***identity matrix*** is defined to be an n x n matrix that has ones on the diagonal and zeros everywhere else.

Matrix inverse

- Let A be a matrix. Then the ***<u>multiplicative inverse</u>*** of A is defined to be the matrix that satisfies

$$A^{-1} \cdot A = \mathbb{I} = A \cdot A^{-1}$$

- Notice, A must be a square matrix, i.e. nxn.
    - If it is not, both multiplications are not defined.

What can we do with an inverse?

- Suppose you have a linear system posed as

$$A \cdot \vec{x} = \vec{b}$$

- Then multiply both sides by the inverse

$$A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$(A^{-1} \cdot A) \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$(\mathbb{I}) \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$\boxed{\vec{x} = A^{-1} \cdot \vec{b}}$$ Solution

Why is this useful?

- If you need to solve a system once, Gaussian Elimination is more efficient than this method.
- If you need to do so many times, this is more efficient.
  - You can precompute A^{-1} once, then multiply by it when needed.

How do we compute and Inverse?

- Let A be an nxn square matrix.
  - Form the augmented matrix $\left[\ A \mid \mathbb{I}\ \right]$
  - Then row reduce to $\left[\ \mathbb{I} \mid A^{-1}\ \right]$

Example

- Compute the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

- Step 1

$$\begin{array}{c} \qquad A \qquad\qquad\qquad \mathbb{I} \\ \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Step 2

- Use ***<u>Gaussian Elimination</u>*** to row reduce the left side.
  - The right side is just along for the ride

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Example Continued

- -2*R1 + R2 ; -2*R1 + R3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

Example Continued

- 2*R2 + R3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Example Continued

- -1 * R1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Example Continued

- 3*R3 + R2 ; -3*R3 + R1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Example Continued

- -2*R2 + R1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Example Continued

$$
\begin{array}{c}
\qquad\qquad \color{blue}{\mathbb{I}} \qquad\qquad\qquad\qquad\qquad \color{blue}{A^{-1}} \\
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -40 & 16 & 9 \\
0 & 1 & 0 & 13 & -5 & -3 \\
0 & 0 & 1 & 5 & -2 & -1
\end{array}\right]
\end{array}
$$

$$
A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}
$$

- You can verify that $A^{-1} \cdot A = A \cdot A^{-1} = \mathbb{I}$

Recap

- Given an nxn matrix A
  - Step 1: Form $[\ A \mid \mathbb{I}\ ]$
  - Step 2 : Reduce the left side to ***reduced row echelon form*** $[\ \mathbb{I} \mid A^{-1}\ ]$
  - Read off the inverse.

Caveats

- Not all matrices are invertible.
- A matrix must be square to be invertible.
- However, even some square matrices are not invertible.
  - If the matrix has no inverse, Gaussian Elimination will simply fail

Example

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

- The left side cannot be turned into the identity!
- So this matrix is not invertible.